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About a Theorem of Paolo Codecà's and Ω-Estimates for Arithmetical Convolutions

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A formula is proved, that can be used to obtain Ω-estimates in a certain class of arithmetical convolutions. It is applied to various functions, in particular to the classical error terms $H(x) := \sum_{n \leqslant x} \phi(n)/n - 6x/\pi^2$, $G(x) := \sum_{n \leqslant x} n/\phi(n) - 315\zeta(3)x/2\pi^4 + \frac{1}{2}\log x$, and $F(x) := \sum_{n \leqslant x} \sigma(n)/n - \pi^2 x/6 + \frac{1}{2}\log x$, where ϕ, σ, and ζ are the Euler, divisor, and Riemann functions. The new estimates $F(x) = \Omega_{\pm}(\log\log x)$ and $G(x) = \Omega_{\pm}(\log\log x)$ are then obtained. As a further application we have $N_H(x) \geqslant (\log\log x)^{3/2}(\log\log\log x)^{-2}$ where $N_H(x)$ denotes the number of changes in sign of the integer-valued function $H(n)$. It is conjectured by Erdös that $N_H(x) \sim cx$.

1. Introduction and Statement of the Results

Consider the class $C(\alpha, f)$ of real-valued functions h defined on $[1, \infty)$ that can be written in the form

$$h(x) = \sum_{n \leqslant x} \frac{\alpha(n)}{n} f(x/n) + o(1) \quad (x \to \infty), \tag{1}$$

where $\alpha(n)$ is a sequence of real numbers satisfying, for some constant K,

$$\sum_{n \leqslant x} \alpha(n) = Kx + o(x) \tag{2}$$

and

$$\sum_{n \leqslant x} |\alpha(n)| = O(x), \tag{3}$$

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and where f is a periodic function of period T, of bounded variation on $[0, T]$, and such that

$$\int_0^T f(u)\,du = 0. \tag{4}$$

If we set

$$H(x) := \sum_{n \leqslant x} \frac{\phi(n)}{n} - \frac{6}{\pi^2}x, \tag{5}$$

$$G(x) := \sum_{n \leqslant x} \frac{n}{\varphi(n)} - \frac{315\zeta(3)}{2\pi^4}x + \frac{\log x}{2} + \frac{\gamma}{2} + 1 + \frac{1}{2}\sum_p \frac{\log p}{p(p-1)}, \tag{6}$$

$$F(x) := \sum_{n \leqslant x} \frac{\sigma(n)}{n} - \frac{\pi^2}{6}x + \frac{\log x}{2} + \gamma/2 + 1, \tag{7}$$

$$P(x) := \sum_{n \leqslant x} \frac{1}{n}\cos(x/n), \tag{8}$$

and

$$Q(x) := \sum_{n \leqslant x} \frac{1}{n}\sin(x/n), \tag{9}$$

where ϕ denotes Euler's function, $\sigma(n)$ the sum of the positive divisors of n, ζ the Riemann zeta function, and γ Euler's constant, then H, G, F, P, and Q all belong to a $C(\alpha, f)$. It is indeed known that

$$H(x) := -\sum_{n \leqslant x} \frac{\mu(n)}{n}\psi(x/n) + o(1), \tag{10}$$

$$G(x) := -\sum_{n \leqslant x} \frac{\mu^2(n)}{\varphi(n)}\psi(x/n) + o(1), \tag{11}$$

and

$$F(x) := -\sum_{n \leqslant x} \frac{1}{n}\psi(x/n) + o(1), \tag{12}$$

where $\psi(y) := \{y\} - \frac{1}{2}$, μ is the Moebius function, and $\{y\}$ denotes the fractional part of y. [The proofs of (10) and (12) are quite straightforward, with the use of the prime number theorem in the case of H. For a proof of (11) see [14, pp. 583–584].]

In the case where a function $g \in C(\alpha, f)$ satisfies the additional condition

$$g(x) = \sum_{n \leqslant z} \frac{\alpha(n)}{n} f(x/n) + K \int_1^{\infty} \frac{f(u)}{u}\, du + o(1), \tag{13}$$

for some positive and strictly increasing function $z = z(x)$ such that

$$z(x) = o(x) \qquad \text{and} \qquad z(x) \to \infty \qquad (x \to \infty), \tag{14}$$

where the constant K is defined in (2), we shall say that $g \in C_z(\alpha, f)$. Codecà [2] proves that if $g \in C_z(\alpha, f)$ for some function z which in addition is *slowly varying* (i.e., such that $z(x) = o(x^{\varepsilon})$ for every positive ε), then the kth asymptotic mean of g,

$$M(g^k) := \lim_{x \to \infty} \frac{1}{x} \int_1^x (g(t))^k\, dt, \tag{15}$$

exists and is finite for each positive integer k. I have proved [10] that this result is applicable to $g = H$, F, P, Q, and $G_{-1,l}$ (the last function is defined in (26) below), and I conjecture that it is also applicable to $g = G$.

[*Note.* Actually, Codecà proves (15) under the stronger hypothesis that the sequence $\alpha(n)$ remains bounded. We leave it to the reader to check that with little change this proof is still valid under the weaker assumption (3).]

But despite their apparent regularity, these functions are actually unbounded; more precisely we know that as $x \to \infty$,

$$H(x) = \Omega(\log\log\log x) \qquad [12], \tag{16}$$

$$H(x) = \Omega_{\pm}(\log\log\log\log x) \qquad [4], \tag{17}$$

$$G(x) = \Omega(\log\log x) \qquad [14], \tag{18}$$

$$F(x) = \Omega_{-}(\log\log x) \qquad [9], \tag{19}$$

$$\limsup_{x \to \infty} F(x) = \infty \qquad [1], \tag{20}$$

$$P(x) = \Omega_{+}(\log\log x) \qquad [5], \tag{21}$$

and

$$Q(x) = \Omega_{\pm}((\log\log x)^{1/2}) \qquad [5]. \tag{22}$$

[Concerning (21) and (22), we note that although Hardy and Littlewood only claim to obtain $P(x) = \Omega(\log\log x)$ and $Q(x) = \Omega((\log\log x)^{1/2})$, what they actually prove is (21) and the Ω_{+}-estimate in (22) [5, Theorems 6 and 7]; moreover, a fairly easy modification in their proof of the latter yields the Ω_{-}-estimate in (22).]

The aim of this paper is to provide a tool (Theorem 1 below) that can be used to seek Ω-estimates for any function $g \in C_z(\alpha, f)$, provided that the function z is increasing, but not too wildly, and satisfies (14). As illustrations we then use Theorem 1 to obtain (17), (18), (21), and (22), and to replace (19) and (20) by

$$F(x) = \Omega_{\pm}(\log\log x), \tag{23}$$

and (18) by

$$G(x) = \Omega_{\pm}(\log\log x). \tag{24}$$

Estimate (23) is equivalent to

$$G_{-1,1}(x) = \Omega_{\pm}(\log\log x), \tag{25}$$

where in general, for a a real number and l a natural number,

$$G_{a,l}(x) := \sum_{n \leqslant \sqrt{x}} n^a \psi_l(x/n), \tag{26}$$

where $\psi_l(y)$ denotes the function $B_l(\{y\})$ and $B_l(u)$ the lth Bernoulli polynomial (for an introduction to these functions, see [13]). The functions $G_{a,l}$ have been extensively studied in the literature (for an up-to-date account, see [11]). As a further application of Theorem 1, we shall prove

$$G_{-1,2l}(x) = \Omega_*(\log\log x) \tag{27}$$

and, for $a < -1$,

$$G_{a,2l}(x) = \Omega_*(1), \tag{28}$$

where $* = (-1)^{l+1}$ is the sign of the $2l$th Bernoulli number B_{2l}. Equations (27) and (28) should be compared with

$$G_{-1,l}(x) = O((\log x)^{2/3}) \qquad (l \geqslant 1) \tag{29}$$

(the case $l = 1$ is treated in [16] and can be generalised) and the trivial

$$G_{a,l}(x) = O(1) \qquad (a < -1, l \geqslant 1). \tag{30}$$

We now state our main result.

THEOREM 1. *Let $A = A(x) > 0$ and $B = B(x) \geqslant 0$ be integer-valued functions, and $z(x)$ be a positive, strictly increasing, continuous and unbounded function. Suppose that z is regularly O-varying, i.e.,*

$$\limsup_{x \to \infty} \frac{z(2x)}{z(x)} =: S < \infty, \tag{31}$$

and that

$$u(x) := z(Ax+B) = o(x) \qquad (x \to \infty). \tag{32}$$

Then, for $g \in C_z(\alpha, f)$, *we have*

$$\frac{1}{x} \sum_{n \leqslant x} g(An+B) = \sum_{k \leqslant u(x)} \frac{\alpha(k)}{k} \left(\frac{1}{k^*} \sum_{n \leqslant k^*} f\left(\frac{n}{k^*} + \frac{B}{k} \right) \right) + E(x), \tag{33}$$

where k^* *denotes* $k/(A, k)$ *and*

$$E(x) = O(1) \qquad (x \to \infty). \tag{34}$$

If in addition at least one of the conditions

$$\begin{gathered} S = 1 \ (z \textit{ is slowly varying}), \\ \sum_{x < n \leqslant x + o(x)} |\alpha(n)| = o(x) \qquad (x \to \infty), \end{gathered} \tag{35}$$

and

$$A(x) = o(z(A\eta x + B)) \qquad \textit{for some} \quad \eta = \eta(x) \to 0 \ (x \to \infty) \tag{36}$$

holds, then we can replace (34) *by the more precise*

$$E(x) = K \int_1^\infty \frac{f(u)}{u} \, du + o(1) \qquad (x \to \infty), \tag{37}$$

with K *as in* (2).

Remark. Theorem 1 generalises Codecà's [1, Theorem 1], which states (33) (with (37)) in the special case where A and B are constants, and under an hypothesis implying that $g \in C_z(\alpha, f)$ for $z = x(\log x)^{-c}$ and some constant $c > 1$.

2. Proof of Theorem 1

We need four lemmata.

Lemma 1. *For any pair of real numbers* a *and* b *such that* $1 \leqslant a \leqslant b$ *we have, with the notation of Theorem* 1,

$$\sum_{a \leqslant n \leqslant b} f\left(\frac{An+B}{k} \right) = \frac{b-a}{k^*} \sum_{n \leqslant k^*} f\left(\frac{n}{k^*} + \frac{B}{k} \right) + O(k^*), \tag{1}$$

where the constant implied by the error term can be chosen so as not to depend on A, B, a, *or* b.

Proof. This follows easily from the fact that if S denotes a complete system of residues to modulus k^*, then

$$\sum_{n \in S} f\left(\frac{An+B}{k}\right) = \sum_{n \leqslant k^*} f\left(\frac{n}{k^*} + \frac{B}{k}\right). \tag{2}$$

LEMMA 2 (Codecà [1, (4.6)]). *With the notation of Theorem 1,*

$$\frac{1}{k^*} \sum_{n \leqslant k^*} f\left(\frac{n}{k^*} + \frac{B}{k}\right) = O\left(\frac{1}{k^*}\right), \tag{3}$$

where again the implied constant depends neither on A nor on B.

LEMMA 3. *If the increasing function z satisfies*

$$\limsup_{x \to \infty} \frac{z(2x)}{z(x)} = 1, \tag{4}$$

then there is a function $\eta = \eta(x) \to 0$ $(x \to \infty)$ such that

$$z(\eta x) = z(x)\ (1 + o(1)) \qquad (x \to \infty). \tag{5}$$

Proof. It follows from (4) that for any positive integer n,

$$\limsup_{y \to \infty} \frac{z(y)}{z(2^{-n}y)} = 1.$$

LEMMA 4. *Let the notation be that of Theorem 1, and set $v(y) = z(Ay + B)$ for $1 \leqslant y \leqslant x$; let $w(k)$ denote the inverse function of v applied to k if $k \geqslant v(1)$, and stand for 1 otherwise. Then*

$$\sum_{k \leqslant u(x)} \frac{(A, k)\,|\alpha(k)|\; w(k)}{k^2} = \begin{cases} O(x) \\ o(x) & \text{if (1.35) or (1.36) holds,} \end{cases} \tag{6}$$

where as before $u(x) = z(Ax + B)$.

Proof. We define, for $x \geqslant 1$,

$$R_i := \left\{k \in \mathbb{N},\ v\left(\max\left(1, \frac{x}{2^i}\right)\right) < k \leqslant v\left(\frac{x}{2^{i-1}}\right)\right\}$$

$$(i = 1, \dots, M := [\log x / \log 2] + 1)$$

and

$$R_{M+1} := \{k \in \mathbb{N},\ k \leqslant v(1)\}.$$

If (1.35) holds, let η be provided by Lemma 3; if (1.36) holds, let η be as in (1.36); otherwise set $\eta = 1$. Also set $N := [-\log \eta/\log 2] + 1$. With these notations the sum to be estimated can be written as

$$\sum_{i=1}^{M+1} \sum_{k \in R_i} \frac{(A, k)\,|\alpha(k)|\,w(k)}{k^2} = \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3, \tag{7}$$

say, where the ranges of summation in $\sum_1$, $\sum_2$, $\sum_3$ are respectively $1 \leqslant i \leqslant N-1$, $N \leqslant i \leqslant M$, $i = M+1$, and where $\sum_1 = 0$ if $\eta = N = 1$. By (1.3), the Euler summation formula, and (1.32), we have

$$\sum\nolimits_3 \leqslant \sum_{k \leqslant v(1)} \frac{|\alpha(k)|}{k} = O(\log x); \tag{8}$$

by using in addition the definition of N and (1.31) we obtain

$$\sum\nolimits_2 \leqslant \sum_{i=N}^{M} \frac{x}{2^{i-1}} \sum_{k \in R_i} \frac{|\alpha(k)|}{k} = O(\eta x). \tag{9}$$

If, besides, (1.35) holds we have

$$\sum\nolimits_1 \leqslant \sum_{i=1}^{N-1} \frac{x}{2^{i-1}} \sum_{k \in R_i} \frac{|\alpha(k)|}{k} = o(x); \tag{10}$$

and if (1.36) holds we have

$$\sum\nolimits_1 \leqslant x \sum_{k > v(\eta x)} \frac{(A, k)\,|\alpha(k)|}{k^2} = o(x), \tag{11}$$

since then for any positive integer n, $nA \leqslant v(\eta x)$ if x is large enough, and since

$$\sum_{k > nA} \frac{(A, k)\,|\alpha(k)|}{k^2} \leqslant A \sum_{k > nA} \frac{|\alpha(k)|}{k^2} = O(1/n).$$

We now pass to the proof of the theorem. If we write

$$g_z(x) := \sum_{n \leqslant z} \frac{\alpha(n)}{n} f(x/n), \tag{12}$$

then, since $g \in C_z(\alpha, f)$ by hypothesis, it is clearly sufficient to show that

$$\begin{aligned} \frac{1}{x} \sum_{n \leqslant x} g_z(An + B) = \sum_{k \leqslant u(x)} \frac{\alpha(k)}{k} \left(\frac{1}{k^*} \sum_{n \leqslant k^*} f\left(\frac{n}{k^*} + \frac{B}{k} \right) \right) \\ + \begin{cases} O(1) & \\ o(1) & \text{if} \quad (1.35) \text{ or } (1.36) \text{ holds.} \end{cases} \end{aligned} \tag{13}$$

The left side of (13) is

$$\frac{1}{x}\sum_{n\leqslant x}\sum_{k\leqslant v(n)}\frac{\alpha(k)}{k}f\left(\frac{An+B}{k}\right)=\frac{1}{x}\sum_{k\leqslant u(x)}\frac{\alpha(k)}{k}\sum_{w(k)\leqslant n\leqslant x}f\left(\frac{An+B}{k}\right),$$

where v and w are as in Lemma 4, By using Lemma 1 we may rewrite the last expression as

$$\frac{1}{x}\sum_{k\leqslant u(x)}\frac{\alpha(k)}{k}\left(\sum_{n\leqslant k^*}f\left(\frac{n}{k^*}+\frac{B}{k}\right)\frac{x-w(k)}{k^*}+O(k^*)\right).$$

And thus

$$\frac{1}{x}\sum_{n\leqslant x}g_z(An+B)=\beta+\delta+\varepsilon, \tag{14}$$

say, where

$$\beta=\sum_{k\leqslant u(x)}\frac{\alpha(k)}{k}\left(\frac{1}{k^*}\sum_{n\leqslant k^*}f\left(\frac{n}{k^*}+\frac{B}{k}\right)\right), \tag{15}$$

where, by (1.3) and (1.32)

$$\varepsilon=\frac{1}{x}\sum_{k\leqslant u(x)}\frac{|\alpha(k)|}{k}O(k^*)=o(1) \tag{16}$$

and where, by Lemma 2,

$$\delta=O\left(\frac{1}{x}\sum_{k\leqslant u(x)}\frac{(A,k)|\alpha(k)|w(k)}{k^2}\right). \tag{17}$$

Lemma 4 concludes the proof.

3. Applications

In this section we sketch the proofs of the Ω-estimates for H, G, F, P, Q, and the $G_{a,l}$ announced in Section 1. We first state a few auxiliary results.

Lemma 5 (Codecà [1, Lemma 1]). *Let z be a positive unbounded function, strictly increasing and such that*

$$\sqrt{x}=o(z(x)) \quad \textit{and} \quad z(x)=o(x) \quad \textit{as} \quad x\to\infty. \tag{1}$$

If $g\in C(1,f)$, then $g\in C_z(1,f)$.

LEMMA 6. *With the notation of Theorem* 1 *and* ψ_l *as in* (1.26), *we have*

$$\frac{1}{k^*}\sum_{n\leqslant k^*}\psi_l\left(\frac{n}{k^*}+\frac{B}{k}\right)=\frac{1}{k^{*l}}\psi_l\left(\frac{B}{(A,k)}\right). \tag{2}$$

Lemma 6 is a straightforward consequence of a well-known property of the Bernoulli polynomials [13, (1.6.1)],

$$B_l(my)=m^{l-1}\sum_{n=0}^{m-1}B_l\left(\frac{n}{m}+y\right). \tag{3}$$

The sine and cosine functions satisfy a similar property: it is easy to see that

LEMMA 7. *With the notation of Theorem* 1 *we have*

$$\sum_{n\leqslant k^*}\begin{matrix}\sin\\ \cos\end{matrix}\left(2\pi\left(\frac{n}{k^*}+\frac{B}{k}\right)\right)=\begin{cases}{}^{\sin}_{\cos}(2\pi B/k) & \text{if } k^*=1\\ 0 & \text{otherwise.}\end{cases} \tag{4}$$

We shall also use

LEMMA 8 (Uchiyama [15]). *There is a positive constant* λ *such that, as* $m\to\infty$,

$$\prod_{\substack{p\leqslant m\\ p\equiv 1(4)}}(1+1/p)\sim\lambda(\log m)^{1/2}. \tag{5}$$

Here and in the sequel, the symbol p exclusively denotes prime numbers.

We now pass to the oscillation estimates.

(a) P (*proof of* (1.21)). *If* A, B, z, and u satisfy the assumptions of Theorem 1 and of Lemma 5, then by Lemmata 5 and 7 and by Theorem 1 we have

$$\frac{1}{x}\sum_{n\leqslant x}P(2\pi(An+B))=\sum_{\substack{k\leqslant u(x)\\ k\mid A}}\frac{1}{k}\cos(2\pi B/k)+O(1) \tag{6}$$

With the special choice

$$A=m!=x^{1/4},\qquad B=0,\qquad z(x)=x^{3/4}, \tag{7}$$

Eq. (6) implies that, as $m\to\infty$,

$$\frac{1}{x}\sum_{n\leqslant x}P(2\pi An)\geqslant\sum_{k\leqslant m}\frac{1}{k}\sim\log\log x, \tag{8}$$

and (1.21) follows.

(b) Q (*proof of* (1.22)). As in the treatment of P, for a suitable choice of the functions involved we have

$$\frac{1}{x}\sum_{n\leqslant x} Q(2\pi(An+B)) = \sum_{\substack{k\leqslant u(x)\\ k\mid A}} \frac{1}{k}\sin(2\pi B/k) + O(1). \tag{9}$$

This time we set

$$A = 4B = x^{1/4}, \qquad B = \prod_{\substack{p\leqslant m\\ p\equiv 1(4)}} p, \qquad z(x) = x^{3/4}, \tag{10}$$

and (9) implies, with Lemma 8, that

$$\frac{1}{x}\sum_{n\leqslant x} Q(2\pi B(4n+1)) \geqslant \frac{1}{4}\sum_{k\mid B}\frac{1}{k} \geqslant \frac{1}{5}\prod_{\substack{p\leqslant m\\ p\equiv 1(4)}} (1+1/p) \sim \frac{\lambda}{5}(\log\log x)^{1/2}, \tag{11}$$

as $m\to\infty$. From (11) we obtain the Ω_+-estimate in (1.22); the choice

$$A = \frac{4}{3}B = x^{1/4} = 4D, \qquad D = \prod_{\substack{p\leqslant m\\ p\equiv 1(4)}} p, \qquad z(x) = x^{3/4} \tag{12}$$

yields similarly the Ω_--estimate.

(c) H (*proof of* (1.17)). Lemma 5 is not applicable in this case; however, as we point out in [10], Walfisz' [16] and Codecà's [2] estimates imply that $H\in C_z(\mu,\psi)$ for any function z satisfying (1.14) and $z_0\leqslant z$ for some slowly varying function z_0, and thus a fortiori for $z(x)=x^{3/4}$. We set

$$A = \prod_{p\leqslant m} p = x^{1/4}, \qquad B = A - m, \qquad z(x) = x^{3/4}, \tag{13}$$

and make use, with Theorem 1, of (1.10) and Lemma 6 to write

$$\begin{aligned}\frac{1}{x}\sum_{n\leqslant x} H(An+B) &= -\sum_{k\leqslant u(x)}\frac{\mu(k)}{k^2}(A,k)(\{B/(A,k)\}-\tfrac{1}{2}) + O(1)\\ &= \sum_{\substack{k\leqslant u(x)\\ (A,k)\mid m}}\frac{\mu(k)}{k^2}(A,k) - \frac{1}{2}\sum_{k\leqslant u(x)}\frac{\mu(k)}{k^2}(A,k)\\ &\quad + \sum_{k\leqslant u(x)}\frac{\mu(k)}{k^2}(A,k)\left\{\frac{m}{(A,k)}\right\} + O(1);\end{aligned}$$

$$\frac{1}{x}\sum_{n\leqslant x} H(An+B) = \Sigma_1 + \Sigma_2 + \Sigma_3 + O(1), \tag{14}$$

say. We have

$$\sum\nolimits_1 = \left(\sum_{k=1}^{m} + \sum_{\substack{k=m+1 \\ (A,k)\mid m}}^{[u(x)]} \right) \frac{\mu(k)}{k^2} (A,k) = \phi(m)/m + O\left(m \sum_{k>m} \frac{1}{k^2} \right);$$

$$\sum\nolimits_1 = O(1). \tag{15}$$

Now

$$-2 \sum\nolimits_2 = \left(\sum_{k=1}^{\infty} - \sum_{k>u(x)} \right) \frac{\mu(k)}{k^2} (A,k) = \sum\nolimits_4 + \sum\nolimits_5, \tag{16}$$

say, where

$$\sum\nolimits_4 = \prod_{p\mid A} (1-1/p) \prod_{p\nmid A} (1-1/p^2) = o(1) \tag{17}$$

as $m \to \infty$, and where $|\sum_5|$ is bounded above by

$$\sum_{k\nmid A} \frac{|\mu(k)|\,(A,k)}{k^2} = \sum_{\substack{k=k_1k_2 \\ (k_1,A)=1 \\ k_1>m;\, k_2\mid A}} \frac{|\mu(k)|}{k_2 k_1^2} = O(\log m/m),$$

as $m \to \infty$. Thus

$$\sum\nolimits_2 = o(1). \tag{18}$$

We turn to $\sum_3$, which we write as

$$\sum_{k=1}^{m} \frac{\mu(k)}{k} \{m/k\} + \sum_{\substack{k=m+1 \\ k\nmid A}}^{[u(x)]} \frac{\mu(k)}{k^2} (A,k)\{m/(A,k)\} + m \sum_{\substack{k=m+1 \\ k\mid A}}^{[u(x)]} \frac{\mu(k)}{k^2}.$$

The first sum is $-H(m)+o(1)$, the second one can be estimated as $\sum_5$ above, and the last one is clearly $O(1)$. Hence

$$\sum\nolimits_3 = -H(m) + O(1). \tag{19}$$

Replacing (15), (18), and (19) in (14) we obtain

$$\frac{1}{x} \sum_{n \leqslant x} H(An+B) = -H(m) + O(1), \qquad 20)$$

which, in view of (1.16) and (13), implies (1.17).

(d) F (*proof of* (1.23)). By using (1.12), Theorem 1, and Lemmata 5 and 6 we have, for suitable A, B, and z,

$$\frac{1}{x}\sum_{n\leqslant x} F(An+B) = -\sum_{k\leqslant u(x)} \frac{(A,k)}{k^2}(\{B/(A,k)\}-1/2)+O(1). \tag{21}$$

If we set

$$A=m!=x^{1/4}, \qquad B=0, \qquad z(x)=x^{3/4}, \tag{22}$$

the right side of (21) is larger than

$$\tfrac{1}{2}\log m+O(1)\sim\frac{1}{2}\log\log x \tag{23}$$

as $m\to\infty$, and similarly if

$$A=m!=x^{1/4}, \qquad B=A-1, \qquad z(x)=x^{3/4}, \tag{24}$$

the right side of (21) is smaller than

$$-\frac{1}{2}\log m+O(1)\sim-\frac{1}{2}\log\log x \tag{25}$$

as $m\to\infty$. From (23) and (25) follows (1.23).

(e) G (*proof of* (1.24)). Lemma 5 is here again not applicable. This time we refer to [14, (4.5)], which implies that $G\in C_z(n\mu^2(n)/\varphi(n), \psi)$ for $z(x)=x^{3/4}$. We use Theorem 1 and Lemma 6 to write

$$\frac{1}{x}\sum_{n\leqslant x} G(An+B) = -\sum_{k\leqslant u(x)} \frac{\mu^2(k)}{k\varphi(k)}(A,k)\left(\left\{\frac{B}{(A,k)}\right\}-\frac{1}{2}\right)+O(1) \tag{26}$$

for

$$A:=\prod_{p\leqslant m} p=x^{1/4} \qquad \text{and} \qquad z(x)=x^{3/4}. \tag{27}$$

As in (d), for the choice $B=0$ (26) is larger than

$$\frac{1}{2}\sum_{k\leqslant m}\frac{\mu^2(k)}{\varphi(k)}+O(1)\sim\frac{1}{2}\log\log x \tag{28}$$

(see [14, (2.2)]), and for $B=A-1$ is smaller than

$$-\frac{1}{2}\sum_{k\leqslant m}\frac{\mu^2(k)}{\varphi(k)}+O(1)\sim-\frac{1}{2}\log\log x \tag{29}$$

(here we use the fact that $\sum \mu^2(k)/k\varphi(k)$ converges). From (28) and (29) follows (1.24).

(f) $G_{a,l}$ *(proof of* (1.27) *and* (1.28)). For suitable A, B, and z (we must ensure this time that (1.36) be satisfied, as we need estimate (1.37)), Theorem 1 and Lemma 6 yield

$$\frac{1}{x} \sum_{n \leqslant x} G_{a,l}(An+B) = \sum_{k \leqslant u(x)} \frac{(A,k)^l}{k^{l-a}} \psi_l(B/(A,k)) + o(1). \tag{30}$$

Condition (1.36) is satisfied with

$$A = m! = x^{1/4}, \qquad B = 0, \qquad z(x) = x^{3/4}, \tag{31}$$

and with this choice the right side of (30) can be written as

$$B_l \sum_{k \leqslant m} k^a + B_l \sum_{k=m+1}^{[u(x)]} \frac{(A,k)^l}{k^{l-a}} + o(1), \tag{32}$$

where B_l is the lth Bernoulli number. Since in (32) all the terms in both sums are of the same sign, we have obtained the desired results.

Remark 1. For $a \leqslant -1$ the function $G_{a,l}$ belongs, by definition, to $C_{\sqrt{x}}(n^{1+a}, \psi_l)$, with the associated constant $K=0$ (see (1.2)). If we consider instead the associated function

$$G^*_{a,l}(x) := \sum_{n \leqslant x} n^a \psi_l(x/n), \tag{33}$$

then it can be shown (for instance by using the method of Walfisz' [16, Chap. III]) that, with $K=1$ if $a=-1$ and $K=0$ if $a<-1$,

$$G^*_{a,l}(x) - G_{a,l}(x) = K \int_1^\infty \frac{\psi_l(u)}{u}\, du + o(1) \qquad (x \to \infty). \tag{34}$$

Thus estimates (1.27), and (1.28) remain valid if we replace $G_{a,l}$ by $G^*_{a,l}$.

Remark 2. In the examples discussed above the function f satisfies some property as in Lemmata 6 and 7. In particular, the Bernoulli polynomial B_l satisfies (3.3), which is usually called a *Kubert identity of order* l:

$$(*l): f(my) = m^{l-1} \sum_{n=0}^{m-1} f\left(\frac{n}{m} + y\right). \tag{35}$$

J. Milnor has characterised all f continuous on (0, 1) satisfying (35) for some complex constant l [6].

In view of Lemma 5 it is easy to prove, by using Theorem 1,

THEOREM 2. *Let $g \in C(1, f)$, with f satisfying a Kubert identity of order l, for some real number l. Suppose in addition that*

$$f(0) \neq 0, \tag{36}$$

and let $*$ denote the sign of $f(0)$. Then

$$g(x) = \Omega_*(\log\log x) \qquad (x \to \infty). \tag{37}$$

4. ON THE PROBLEM OF ESTIMATING THE NUMBER OF CHANGES IN SIGN OF H

In this section we use freely the letter c to denote various positive constants.

The function $H(t)$ decreases linearly by $6/\pi^2$ on each interval $[n, n+1)$ and jumps by $+\phi(n)/n$ at $t = n$, where n is a positive integer. Although we know [7] that

$$X_H(x) = cx + o(x) \qquad (x \to \infty), \tag{1}$$

where $X_g(x)$ denotes the number of changes in sign of the function $g(t)$ ($t \in \mathbb{R}$) on the interval $[1, x)$, the problem of estimating the function $N_g(x)$, that counts only the changes in sign of the restricted $g(n)$ ($n \in \mathbb{N}$) in the same interval, seems to be generally more difficult. Erdös conjectures [3] that

$$N_H(x) = cx + o(x) \qquad (x \to \infty), \tag{2}$$

but it is only known to date [8] that

$$N_H(x) \geqslant c \log\log x + O(1) \qquad (x \to \infty). \tag{3}$$

This can be somewhat improved with the help of Theorem 1, and we shall briefly indicate how. We obtain

$$N_H(x) \geqslant c(\log\log x)^{3/2}(\log\log\log x)^{-2} + O(1) \qquad (x \to \infty). \tag{4}$$

We use the fact [16, 10] that $H \in C_z(\mu, \psi)$ for $z(x) = \exp(c(\log x)^{2/3}(\log\log x)^{4/3})$. It follows that Theorem 1 can be applied as was shown in Section 3, but with the choice

$$A = x^{c(\log x)^{1/2}}(\log\log x)^{-2}. \tag{5}$$

The method of [8], which yields (3) when applied to the choice

$$A = x^{1/2} \tag{6}$$

(which is permitted by the main result in [4]), is now applicable to (5) with a few modifications, and leads to (4).

Other results of the same type can be obtained in a similar way, as for instance

$$N_F(x) \geqslant c(\log\log x)^{3/2}(\log\log\log x)^{-1/2} + O(1) \qquad (x \to \infty), \tag{7}$$

which improves [8]

$$N_F(x) \geqslant c \log\log x + O(1) \qquad (x \to \infty). \tag{8}$$

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